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Lorentz Transformation on the Lattice[1]

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We study the possibility of representing the kinematical variables of a free particle in terms of scale factors and integers. The action of a set of transformations from the Lorentz group parametrized by integers on this system of variables are investigated, and it is shown that one can effectively characterize these symmetries on a lattice in this way. By taking the scales sufficiently small, one can arbitrarily closely approach the continuous case.

1. INTRODUCTION

The introduction of a discrete space-time lattice has been useful in the study of quantum gauge field theory. The question of the realization of the continuous symmetries on the lattice, such as the Lorentz group, and the passage to the continuous limit has not yet been completely worked out.

In this paper it is shown that the continuous symmetry group of a free relativistic particle can be approximated on a discrete lattice in such a way that the main features of the group structure are maintained and the discrete realization approaches the original continuous group in the limit that the lattice approaches the continuum.

This is done by extracting a scale and integer coefficients from the physical coordinate and momentum and specifying the transformation properties of these factors.

We have shown that only in the simplest case, the one-dimensional nonrelativistic Galilean group, can one perform a discretization of the values of momentum and position of the free particle, assuming that the corresponding scale factors stay unchanged under transformations from one admissible classical Galilean frame to another.

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To treat successfully the cases of two-dimensional Lorentz transformations and two-dimensional rotations we introduce transformations of the scales in a definite way.

In this way we select a discrete set of admissible states and a discrete set of admissible transformations that are parametrized by integer parameters in such a manner that in the limit of small scale parameters we can arbitrarily closely approach the continuous case. This procedure can be extended to the four-dimensional Minkowski space.

The paper is organized as follows: In Section 2 we consider a one-dimensional nonrelativistic free particle motion and perform a discretization of the values of kinematical variables. This can be done maintaining the scale factors as invariants of the transformations.

We devote Section 3 to the case of two-dimensional Lorentz transformations and its discretization. This is done by allowing the scale factors to transform under Lorentz transformations. We show that proposed discretization procedure coincides with that considered in Section 2 in the nonrelativistic limit. As a technical result of this section we mention the decomposition of the two-dimensional Lorentz transformations into a product of two transformations, each of which forms a group with the usual relativistic law of velocity composition.

In Section 4 we consider a discretization procedure in the case of two-dimensional relativistic free particle motion parametrized by a proper time.

In Section 5, we study the discretization procedure for the compact two-dimensional rotation group in a way similar to the treatment of noncompact Lorentz transformations in Section 2.

And finally in Section 6 we study the general case of the discretization procedure for the $O(3,1)$ Lorentz group.

2. DISCRETIZATION OF THE CLASSICAL ONE-DIMENSIONAL GALILEAN FREE PARTICLE MOTION

Let us first consider a classical Galilean free-particle motion taking place along the x axis of a fixed reference frame L_0. If now v_0 is the velocity of the particle relative to this frame then the momentum p_0 and the energy E_0 of this particle in this frame are given by

$$p_0 = Mv_0, \qquad E_0 = \frac{p_0^2}{2M} \tag{1}$$

where M is the mass of the particle.

We suppose further that at a moment t_0 the position of the particle in this frame L_0 was x_0. If now τ is a "proper time" parametrizing the free motion of the particle then we can talk about the initial position in four-dimensional phase space of a particle at a moment $\tau = 0$: (E_0, p_0, x_0, t_0). At the arbitrary moment τ the position of the particle in phase space is $(E(\tau), p(\tau), x(\tau), t(\tau))$, where

$$E(\tau) = E_0, \qquad p(\tau) = p_0 \tag{2a}$$

and

$$x(\tau) = x_0 + \frac{p_0}{M}\tau, \qquad t(\tau) = t_0 + \tau \tag{2b}$$

Let us consider now an arbitrary reference frame L moving with a velocity v relative to L_0 along the x axis direction common to both L_0 and L reference frames. Then the velocity v' of the above considered particle in this reference frame L is

$$v' = v_0 + v \tag{3}$$

so that the initial (and conserved in time) momentum p'_0 and energy E'_0 are given in L by

$$p'_0 = p'(\tau) = Mv', \qquad E'_0 = E'(\tau) = \frac{Mv'^2}{2} \tag{4}$$

where $p'(\tau)$ and $E'(\tau)$ are momentum and energy at a "proper-time" moment τ. The initial position x'_0 and time t'_0 at a moment $\tau = 0$ are

$$x'_0 = x_0 + vt_0 \tag{5a}$$

$$t'_0 = t_0 \tag{5b}$$

so that the trajectory in phase space of a particle parametrized by τ is given in an arbitrary reference frame L' by (4) and

$$x'(\tau) = x'_0 + \frac{p'}{M}\tau = x_0 + vt_0 + \frac{p'}{M}\tau \tag{6a}$$

$$t'(\tau) = t'_0 + \tau = t_0 + \tau \tag{6b}$$

Let us suppose now that there exist scales for all dimensional quantities entering in (4) and (6). We can choose three independent elementary scales: elementary scale of mass μ, elementary scale for position a, and elementary scale for velocity c.

Then the scales for momentum μ_p and energy μ_ϵ can be written correspondingly as

$$\mu_p = \mu c, \qquad \mu_\epsilon = \frac{\mu^2 c^2}{2M} \tag{7}$$

and the scale for time a_t as

$$a_t = \frac{a}{c}\frac{M}{\mu} \tag{8}$$

We can now turn to the procedure for discretization of the description of the evolution of the free particle in classical one-dimensional nonrelativistic case. For this purpose we suppose that all physical quantities considered above characterizing the free-particle motion can take only admissible values which can be written as products of corresponding elementary scales and integer numbers (we write a caret on top of the corresponding letters standing for these integers).

First of all we rewrite in this way the mass of the particle M, the velocity V_0 of the particle, the velocity of the Galilean transformation V, and the initial position of the particle x:

$$M = \mu \hat{m} \tag{9a}$$

$$V_0 = \frac{c}{\hat{m}}\hat{k}_0, \qquad V = \frac{c}{\hat{m}}\hat{k} \tag{9b}$$

$$x = a\hat{x} \tag{9c}$$

Then we have that

$$V' = \frac{c}{\hat{m}}(\hat{k}_0 + \hat{k}) \equiv \frac{c}{\hat{m}}\hat{k}' \tag{10a}$$

$$P' = \mu_p \hat{k}' \tag{10b}$$

$$E' = \mu_\epsilon \hat{k}'^2 \tag{10c}$$

$$t'(\tau) = a_t(\hat{t}_0 + \hat{\tau}) \tag{10d}$$

$$X'(\tau) = a(\hat{x} + \hat{k}\hat{t}_0 + \hat{k}'\hat{\tau}) \tag{10e}$$

are also represented in "admissible" way that is, as a product of corresponding scales and integer numbers.

Let us note that we have considered up to now only the case of one-dimensional classical Galilean motion of a free particle. In this case all elementary scales stay unchanged under the action of Galilean transformations. As we shall see this is not the case when two-dimensional Lorentz transformations are considered.

3. TWO-DIMENSIONAL LORENTZ TRANSFORMATIONS ON THE LATTICE

Our main goal will be now the generalization of the simple results of the previous section to the case of relativistic free-particle motion. We suppose that as in the classical nonrelativistic case we can limit ourselves to the discrete values of corresponding physical quantities describing a free-particle motion by the introduction of independent elementary scales and representing all physical quantities by products of corresponding scales by integer numbers. As before we distinguish the integer-valued quantities by carets on top of the corresponding letters.

But now we cannot guarantee that these scale factors stay invariant under the Lorentz transformations. In fact as we shall see the action of Lorentz transformations on the momentum and position vectors written as products of corresponding scales and integer numbers, factorizes into action on the scales and on these integers.

As for the Lorentz transformations that we admit, they are selected from the Lorentz group transformations by a procedure we are going to discuss now. The way we introduce what we shall call "admissible transformations" also makes clear the decomposition of kinematical physical quantities of the free particle into products of scales and integers.

Let us suppose now that we are given in some Lorentz frame of reference two-dimensional momentum (p_0, p_1) of the particles moving relative to this frame L_0 so that

$$P_0 = \mu \hat{p}_0, \qquad P_1 = \mu \hat{p}_1 \tag{11}$$

where $\mu > 0$ is some scale parameter having the dimension of momentum, common for p_0 and p_1 decompositions and $\hat{p}_0$ and $\hat{p}_1$ some integer dimensionless numbers such that $\hat{p}_0 > |\hat{p}_1|$, $\hat{p}_0 > 0$. We suppose first that also $\hat{p}_1 > 0$.

The transformation of the momentum in the two-dimensional case is given by

$$p_0' = \frac{p_0 + \beta p_1}{(1-\beta^2)^{1/2}}, \qquad p_1' = \frac{p_1 + \beta p_0}{(1-\beta^2)^{1/2}} \qquad \left(\beta = \frac{V}{c}\right) \tag{12}$$

which can be written taking into account (11) as

$$p'_0 = \mu \left(\frac{1-\beta}{1+\beta} \right)^{1/2} \frac{\hat{p}_0 + \beta \hat{p}_1}{1-\beta} \tag{13a}$$

$$p'_1 = \mu \left(\frac{1-\beta}{1+\beta} \right)^{1/2} \frac{\hat{p}_1 + \beta \hat{p}_0}{1-\beta} \tag{13b}$$

Let us suppose now that in the new frame of reference L the momentum p'_μ can be written as

$$p'_0 = \mu' \hat{p}'_0 \equiv \mu' (\hat{p}_0 + \hat{k}_0) \tag{14a}$$

$$p'_1 = \mu' \hat{p}'_1 \equiv \mu' (\hat{p}_1 + \hat{k}_1) \tag{14b}$$

where μ' is the new scale parameter corresponding to the new Lorentz frame and L and $\hat{k}_0, \hat{k}_1$ are positive integer numbers.

Let us suppose that

$$\mu' = \left(\frac{1-\beta}{1+\beta} \right)^{1/2} \mu \tag{15}$$

Then from (13) and (14) it follows that

$$\frac{\hat{p}_0 + \beta \hat{p}_1}{1-\beta} = \hat{p}_0 + \hat{k}_0 \tag{16a}$$

$$\frac{\hat{p}_1 + \beta \hat{p}_0}{1-\beta} = \hat{p}_1 + \hat{k}_1 \tag{16b}$$

From (16a) it follows that

$$\beta = \frac{\hat{k}_0}{\hat{p}_0 + \hat{p}_1 + \hat{k}_0} \tag{17a}$$

and from (16b) that

$$\beta = \frac{\hat{k}_1}{\hat{p}_0 + \hat{p}_1 + \hat{k}_1} \tag{17b}$$

We can satisfy (17) if we suppose that

$$\hat{k}_0 = \hat{k}_1 \equiv \hat{k} \tag{18}$$

So that in the new Lorentz frame L, moving with velocity $V_k = \beta_k C$ relative to the fixed Lorentz frame L_0, such that

$$\beta_k = \frac{\hat{k}}{\hat{p}_0 + \hat{p}_1 + \hat{k}} < 1 \tag{19}$$

the momentum is given by

$$p_0' \equiv \mu' \hat{p}_0' = \mu'(\hat{p}_0 + \hat{k}) \tag{20a}$$

$$p_1' \equiv \mu' \hat{p}_1' = \mu'(\hat{p}_1 + \hat{k}) \tag{20b}$$

and the momentum scale factor is given by

$$\mu' = \left(\frac{1-\beta_k}{1+\beta_k}\right)^{1/2} \mu \tag{21}$$

where $\hat{k}$ is an arbitrary positive integer number.

Note that admissible $\beta_k < 1$ can take only discrete values [see (19)] and in the case of sufficiently small $\mu(\mu \to 0)$ such that $\hat{p}_0 + \hat{p}_1$ is very large β_k falls almost on the continuum.

The transformations

$$p_0' = \frac{p_0 + \beta p_1}{1-\beta} \tag{22a}$$

$$p_1' = \frac{p_1 + \beta p_0}{1-\beta} \tag{22b}$$

form a group with a usual relativistic law of velocity composition.

Indeed the composition of any two such transformations (22) leads, as it is easy to see, to the transformation of the same kind with the usual relativistic law of the velocity composition

$$\beta_{12} = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \tag{23}$$

and in this set of transformations there exists a unit transformation corresponding to $\beta = 0$ and for any transformation characterized by β there exists an opposite transformation with $\beta' = -\beta$. It is easy to see also that the transformations

$$\mu' = \left(\frac{1-\beta}{1+\beta} \right)^{1/2} \mu \tag{24}$$

also forms a group with the same relativistic velocity addition law (23) as before.

Indeed we have that [see (15)] from

$$\mu' = \left(\frac{1-\beta_1}{1+\beta_1} \right)^{1/2} \mu, \qquad \mu'' = \left(\frac{1-\beta_2}{1+\beta_2} \right)^{1/2} \mu' \tag{25}$$

follows that

$$\mu'' = \left(\frac{1-\beta_{12}}{1+\beta_{12}} \right)^{1/2} \mu \tag{26}$$

where β_{12} is given by (23), and the unit and opposite elements are given as in the previous case.

The transformation from one admissible Lorentz frame L, that is, one in which the momentum can be written in the form (11), to some other L where momentum is given by (14), is characterized by admissible relative velocity $v \doteq \beta_{k_1} C$, where

$$\beta_{k_1} = \frac{\hat{k}_1}{\hat{p}_0 + \hat{p}_1 + \hat{k}_1} \tag{27}$$

Then the admissible Lorentz transformation from the Lorentz frame L' to some admissible Lorentz frame L'' is characterized by β_{k_2} such that [see (14)]

$$\beta_{k_2} = \frac{\hat{k}_2}{\hat{p}'_0 + \hat{p}'_1 + \hat{k}_2} = \frac{\hat{k}_2}{\hat{p}_0 + \hat{p}_1 + 2\hat{k}_1 + \hat{k}_2} \tag{28}$$

Since this transformation is parametrized by β_{k_2} which is dependent on the parameter $\hat{k}_1$, characterizing the previous transformation from L_0 to L' we conclude that the second admissible transformation in the composition of two admissible transformations depends on the previous one. This means that the composition of any two admissible transformations is not necessarily itself an admissible transformation. This in turn means that the set of admissible transformations which is a subset of the group of all transformations (22) does not itself form a group.

But the composition of a "chain" type characterized by β_{k_1} and β_{k_2} [see (27) and (28)] (where β_{k_2} depends on the previous transformation parameter $\hat{k}_1$) is once again an admissible transformation characterized by $\beta_{k_1k_2}$ where [because of (23)]

$$\beta_{k_1k_2}=\frac{\beta_{k_1}+\beta_{k_2}}{1+\beta_{k_1}\beta_{k_2}}=\frac{\hat{k}_{12}}{\hat{p}_0+\hat{p}_1+\hat{k}_{12}} \tag{29}$$

so that

$$\hat{k}_{12}=\hat{k}_1+\hat{k}_2 \tag{30}$$

We have been dealing up to now with the set of admissible [compatible with equation (19)] Lorentz transformations leaving invariant the positive sign of the spacelike component of the momentum. This means that in all admissible frames of reference considered until now the movement of the particle takes place in the positive x direction. Yet we can consider the systems of reference possessing negative relative velocities β_{-k}, which can be written in the form

$$\beta_{-k}=\frac{-\hat{k}}{\hat{p}_0+\hat{p}_1-\hat{k}} \tag{31}$$

To clarify this point we note that in an arbitrary admissible Lorentz frame considered until now we have that

$$\hat{p}_0=\hat{m}+\hat{k}_0, \qquad \hat{p}_1=\hat{k}_0 \tag{32}$$

where $\hat{m}$ corresponds to the mass of the particle. We suppose that the scale parameter in the system of rest of the particle is μ_0 then

$$M=\mu_0\hat{m} \tag{33}$$

where M is the mass of the particle.

Then the admissible transformations to the Lorentz frames moving forward relative to the fixed frame characterized by the integer number $\hat{k}_0$ [see (32)] must possess negative velocity β equal to

$$\beta=-\frac{\hat{k}}{\hat{p}_0+\hat{p}_1-\hat{k}}=-\frac{\hat{k}}{\hat{m}+2\hat{k}_1-\hat{k}} \tag{34}$$

where $\hat{k}\leqslant\hat{k}_0\hat{k}_0>0$.

Note that transformation with β equal to

$$\beta = -\frac{\hat{k}_0}{\hat{m} + \hat{k}_0} \tag{35}$$

corresponds to the transformation to the system of rest of the particle when

$$\hat{p}_0 = \hat{m} + \hat{k}_0 - k_0 = \hat{m}, \qquad \hat{p}_1 = \hat{k}_0 - \hat{k}_0 = 0 \tag{36}$$

Now let us look for the set of Lorentz transformations leaving invariant the decomposition of the particle momentum

$$p_0 = \mu \hat{p}_0, \qquad p_1 = \mu \hat{p}_1 \tag{37}$$

where $p_0 > 0$ and $p_1 < 0$, so that in all admissible Lorentz frames of this set the movement of the particle takes place along the negative x direction of these Lorentz frames.

For this purpose let us rewrite the usual Lorentz transformation (12) in the following form [compare with (13)]:

$$p_0' = \mu \left(\frac{1+\beta}{1-\beta} \right)^{1/2} \frac{\hat{p}_0 + \beta \hat{p}_1}{1+\beta} \tag{38a}$$

$$p_1' = \mu \left(\frac{1+\beta}{1-\beta} \right)^{1/2} \frac{\hat{p}_1 + \beta \hat{p}_0}{1+\beta} \tag{38b}$$

If we now suppose that after the transformation the new scale parameter μ' is equal to

$$\mu' = \left(\frac{1+\beta}{1-\beta} \right)^{1/2} \mu \tag{39}$$

and that

$$p_0' = \mu'(\hat{p}_0 + \hat{k}) \tag{40a}$$

$$p_1' = \mu'(\hat{p}_1 - \hat{k}) \tag{40b}$$

we come to the admissible velocity parameter β_k equal to

$$\beta_k = -\frac{\hat{k}}{\hat{p}_0 - \hat{p}_1 + \hat{k}} = -\frac{\hat{k}}{\hat{p}_0 + |\hat{p}_1| + \hat{k}} \tag{41}$$

where $\hat{k}$ is an arbitrary positive integer number.

As before it it easy to see that the "chain type" composition analogous to (27), (28) of two admissible transformations of the type now considered also leads to a transformation of this type with the usual relativistic velocity addition law.

Note that for positive integer $\hat{k}$, β_k given by (41) is negative. But transformations with positive β, corresponding to the negative integer $\hat{k}$, also belong to this set. The only restriction for the positive β is that it must not be too big so that the corresponding transformation leads to the consideration of the admissible Lorentz frame with respect to which the movement of the particle still takes place in the negative x direction.

To specify this point let us suppose that in some fixed admissible Lorentz frame of the set we are discussing now the momentum of the particle is (p_0, p_1). If μ is the scale of the momentum in this frame then we can write that

$$p_0 = \mu(\hat{m} + \hat{k}_0) \equiv \mu \hat{p}_0 \tag{42a}$$

$$p_1 = -\mu \hat{k}_0 \equiv \mu \hat{p}_1 \tag{42b}$$

where $\hat{m}$ is the same as in (33).

From this fixed frame we can transform to the set of admissible Lorentz frames of reference, characterizing by the relative velocity parameter β_k, where

$$\beta_k = -\frac{\hat{k}}{\hat{p}_0 - \hat{p}_1 + \hat{k}} \tag{43}$$

Taking into account (42) we have that

$$\beta_k = -\frac{\hat{k}}{\hat{m} + 2\hat{k}_0 + \hat{k}} \tag{44}$$

Then the maximum positive relative velocity given by (44), corresponding to the negative $\hat{k}$ equal to $-\hat{k}_0$ corresponds to the transition to the system of rest of the particle. The case when

$$-\hat{k}_0 \leqslant \hat{k} \leqslant 0 \tag{45}$$

corresponds to the transition from the fixed admissible Lorentz frame from the set now considered to the others from the same set moving with respect to this system in the negative x direction, but with respect to which the movement of the particle also takes place in the negative x direction. Note that there is no restriction for the positive values of the integer $\hat{k}$ entering into (44) and that

$$-1<\beta_k \leqslant 0 \tag{46}$$

for

$$0 \leqslant \hat{k} < \infty$$

Let us compare now the two possible sets of transformations considered above. As we have chosen the positive x direction in an arbitrary way let us check if there is a correspondence between the admissible movement of the particle in the positive x direction, described by the first set, and the description of the particle's admissible movement in the negative x direction given by the second set.

It is easy to see that if the free particle possesses the admissible momentum p_μ given by

$$p_0=\mu(\hat{m}+\hat{k}_0), \qquad p_1=\mu\hat{k}_0 \tag{47}$$

then the momentum p'_μ

$$p'_0=\mu(\hat{m}+\hat{k}_0), \qquad p'_1=-\mu\hat{k}_0 \tag{48}$$

corresponding to the particle moving with the same velocity but in the opposite direction is also admissible and that in both cases the scale parameter μ entering in (47) and (48) and equal [see (24) and (39)]

$$\mu=\left(\frac{1-|\beta_k|}{1+|\beta_k|}\right)^{1/2}\mu_0 \tag{49}$$

is the same. Here in (45) β is the velocity of the particle and μ_0 is the scale parameter in the system of rest of the particle, the same as in (33).

The admissible values of the velocity parameter of the particle entering in (49) are

$$\beta_k=\pm\frac{\hat{k}}{\hat{m}+\hat{k}} \tag{50}$$

where $\hat{k}$ is an arbitrary positive integer number and $\hat{m}$ is the same as in (33).

At the end if we compare (19) and (41) giving up to a sign the same values for the admissible relative velocity parameters of Lorentz frames with respect to the fixed frames where the particle is characterized by the momentum (p_0, p_1) and $(p_0, -p_1)$, we come to the conclusion that there is a full correspondence of the description of the particle's movement in the positive x direction given by the first set and its movement in the negative x direction described by the second set.

There exists one special Lorentz frame which can be singled out—the system of rest of the particle which is an admissible Lorentz frame for both of the above discussed sets. This fact makes it possible to join two sets considered above into one single joint set describing the movement of the free particle in both x directions. The admissible relative velocity can be given now in general as a relativistic sum of two admissible velocities corresponding to the same or to two different sets considered above—one of these velocities corresponds to the transition to the system of rest of the particle while the other corresponds to the transition from the system of rest of the particle to the new admissible frame of reference from the same or different set of admissible transformations.

Let us mention also that, as follows from (32) and (42), the following expression

$$\hat{p}_0 - |\hat{p}_1| = \hat{m} \tag{51}$$

stays invariant under the transformations from the joint set.

So we can conclude that the same way as two-dimensional Lorentz transformations leave invariant the mass shell given by a hyperbola, in the case of integers parametrizing possible values of admissible momentum they lie on the "cone" plotted in Figure 1.

Let us suppose now that along with the momentum of the particle (p_0, p_1), where

$$p_0 = \mu \hat{p}_0, \qquad p_1 = \mu \hat{p}_1 \tag{52}$$

given in some admissible Lorentz frame we are given also the position of the particle (x_0, x_1) in the same frame. And let us assume that it can be written in the following form:

$$x_0 = a(\hat{p}_0\hat{q}_0 + \hat{p}_1\hat{q}_1) \tag{53a}$$

$$x_1 = a(\hat{p}_0\hat{q}_1 + \hat{p}_1\hat{q}_0) \tag{53b}$$

The parameter a entering into (53) is the scale parameter having the dimension of length specific to the Lorentz frame considered and $\hat{q}_0, \hat{q}_1$ are arbitrary dimensionless integer numbers.

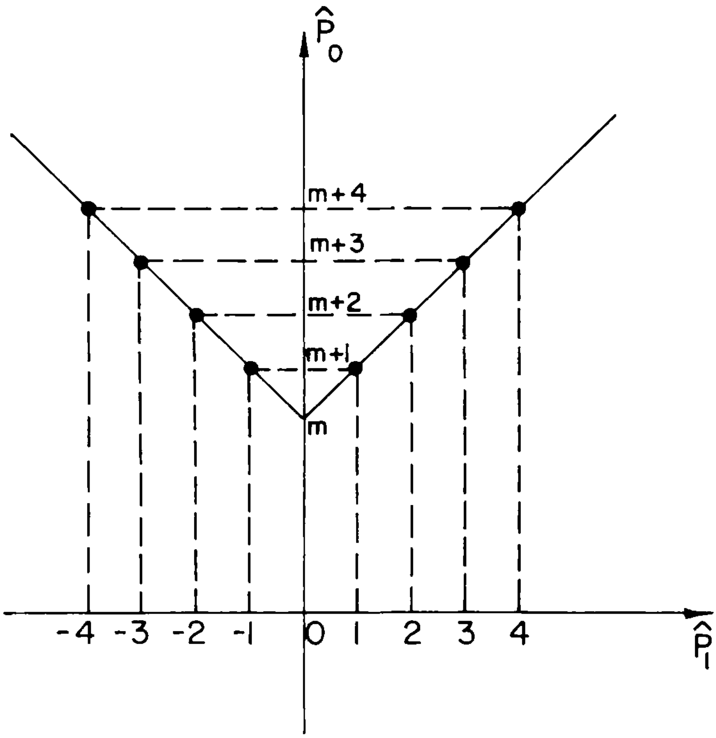

Fig. 1. The "cone" of admissible momentum, replacing the usual mass shell.

We are going to show that the decomposition of the position and time variables into some scale parameter multiplied by integer numbers given by (53) stays invariant under the admissible transformations from the "joint" set of all admissible transformations.

Let us suppose first of all that $p_1 \geqslant 0$. Then the admissible relative velocities characterizing the transformations to the possible Lorentz frames in respect to which the movement of the particle takes place in the positive x direction are given by

$$\beta_k = \frac{\hat{k}}{\hat{p}_0 + \hat{p}_1 + \hat{k}} \tag{54}$$

where $\hat{k}$ is an integer and

$$-\hat{p}_1 \leqslant \hat{k} < \infty$$

In the new admissible frame of reference after the transformation characterized by β_k the momentum is

$$p_0' = \mu' \hat{p}_0' = \mu'(\hat{p}_0 + \hat{k}) \tag{55a}$$

$$p_1' = \mu' \hat{p}_1' = \mu' \hat{k} \tag{55b}$$

As to the position transformation given by

$$x_0' = \frac{x_0 + \beta_k x_1}{(1-\beta_k^2)^{1/2}}, \qquad x_1' = \frac{x_1 + \beta_k x_0}{(1-\beta_k^2)^{1/2}} \tag{56}$$

we can rewrite it in the following form using (53):

$$x_0' = a\left(\frac{1-\beta_k}{1+\beta_k}\right)^{1/2}\left[\left(\frac{\hat{p}_0 + \beta_k \hat{p}_1}{1-\beta_k}\right)\hat{q}_0 + \left(\frac{\hat{p}_1 + \beta_k \hat{p}_0}{1-\beta_k}\right)\hat{q}_1\right] \tag{57a}$$

$$x_1' = a\left(\frac{1-\beta_k}{1+\beta_k}\right)^{1/2}\left[\left(\frac{\hat{p}_0 + \beta_k \hat{p}_1}{1-\beta_k}\right)\hat{q}_1 + \left(\frac{\hat{p}_1 - \beta_k \hat{p}_0}{1-\beta_k}\right)\hat{q}_0\right] \tag{57b}$$

But [see (16)]

$$\frac{\hat{p}_0 + \beta_k \hat{p}_1}{1-\beta_k} \equiv \hat{p}_0' = \hat{p}_0 + \hat{k} \tag{58a}$$

$$\frac{\hat{p}_1 + \beta_k \hat{p}_0}{1-\beta_k} \equiv \hat{p}_1' = \hat{p}_1 + \hat{k} \tag{58b}$$

So if we suppose now that the position scale parameter in new admissible frame is given by

$$a' = \left(\frac{1-\beta_k}{1+\beta_k}\right)^{1/2} a \tag{59}$$

from (57) and (58) we have that

$$x_0' = a'(\hat{p}_0'\hat{q}_0 + \hat{p}_1'\hat{q}_1) \tag{60a}$$

$$x_1' = a'(\hat{p}_0'\hat{q}_1 + \hat{p}_1'\hat{q}_0) \tag{60b}$$

The case when $p_1 \leq 0$ can be treated analogously. Indeed if we are given admissible momentum (p_0, p_1) and position (x_0, x_1) of the free particle moving in the negative x direction ($p_1 < 0$) then the admissible transformations are characterized by relative velocity parameter β_k

$$\beta_k = -\frac{\hat{k}}{\hat{p}_0 - \hat{p}_1 + \hat{k}} = -\frac{\hat{k}}{\hat{p}_0 + |\hat{p}_1| + \hat{k}} \tag{61}$$

where $\hat{k}$ is an integer and

$$-|\hat{p}_1| \leqslant \hat{k} < \infty \tag{62}$$

If we suppose now that as before momentum and position of the particle in some Lorentz frame can be written as

$$p_0 = \mu \hat{p}_0 \tag{63a}$$

$$p_1 = \mu \hat{p}_1 \tag{63b}$$

$$x_0 = a(\hat{p}_0 \hat{q}_0 + \hat{p}_1 \hat{q}_1) \tag{63c}$$

$$x_1 = a(\hat{p}_0 \hat{q}_1 + \hat{p}_1 \hat{q}_0) \tag{63d}$$

then after the admissible transformation characterized by β_k given by (61) momentum is given by

$$p_0' = \mu' \hat{p}_0' = \mu'(\hat{p}_0 + \hat{k}) \tag{64a}$$

$$p_1' = \mu' \hat{p}_1' = \mu'(\hat{p}_1 - \hat{k}) \tag{64b}$$

where

$$\mu' = \left(\frac{1+\beta_k}{1-\beta_k}\right)^{1/2} \mu \tag{65}$$

The transformation of the position given by (56) can be rewritten now as [see (63), (64)]

$$\begin{aligned} x_0' &= a\left(\frac{1+\beta_k}{1-\beta_k}\right)^{1/2}\left[\left(\frac{\hat{p}_0 + \beta_k \hat{p}_1}{1+\beta_k}\right)\hat{q}_0 + \left(\frac{\hat{p}_1 + \beta_k \hat{p}_0}{1+\beta_k}\right)\hat{q}_1\right] \\ &= a'\left[(\hat{p}_0 + \hat{k})\hat{q}_0 + (\hat{p}_1 - \hat{k})\hat{q}_1\right] \equiv a'(\hat{p}_0' \hat{q}_0 + \hat{p}_1' \hat{q}_1) \end{aligned}$$

$$x_1' = a\left(\frac{1+\beta_k}{1-\beta_k}\right)^{1/2}\left[\left(\frac{\hat{p}_0 + \beta_k \hat{p}_1}{1+\beta_k}\right)\hat{q}_1 + \left(\frac{\hat{p}_1 + \beta_k \hat{p}_0}{1+\beta_k}\right)\hat{q}_0\right] \tag{66a}$$

$$= a'\left[(\hat{p}_0 + \hat{k})\hat{q}_1 + (\hat{p}_1 - \hat{k})\hat{q}_0\right] = a'(\hat{p}_0' \hat{q}_1 + \hat{p}_1' \hat{q}_0) \tag{66b}$$

where

$$a' = a\left(\frac{1+\beta_k}{1-\beta_k}\right)^{1/2} = a\left(\frac{1-|\beta_k|}{1+|\beta_k|}\right)^{1/2} \tag{67}$$

So we see that the two special cases considered allow us to take the joint set discussed above as the set of all possible admissible transformations of momentum and position vectors of the free relativistic particle leaving invariant the decomposition given by (52) and (53).

Let us note that the scale parameter a corresponding to the movements of a particle with the same admissible velocities in opposite x directions, is the same and equal to

$$a = \left(\frac{1-|\beta_k|}{1+|\beta_k|}\right)^{1/2} a_0, \qquad \beta_k = \pm\frac{\hat{k}}{\hat{m}+\hat{k}} \tag{68}$$

We mention also that the time and position of the particle in all admissible Lorentz frames given by (53) are characterized by the same integer numbers q_0 and q_1. If we make now the transformation to the system of rest of the particle then from (60) and (36) we have that

$$x_0 = a_0 \hat{m} \hat{q}_0 \tag{69a}$$

$$x_1 = a_0 \hat{m} \hat{q}_1 \tag{69b}$$

which means that $\hat{q}_0$ parametrizes the time and $\hat{q}_1$ the position of the particle in its rest frame. Note also that

$$\hat{p}_0 - |\hat{p}_1| = \hat{m} \tag{70a}$$

$$\frac{\mu}{a} = \frac{\mu_0}{a_0} \tag{70b}$$

and integers $\hat{q}_0$ and $\hat{q}_1$ are invariants of the transformation from the joint set.

4. DISCRETIZATION OF RELATIVISTIC GALILEAN FREE-PARTICLE MOTION

We suppose that in some Lorentz frame L_0 is given an initial phase-space point $(p_0^0, p_1^0, x_0^0, x_1^0)$ corresponding to the initial momentum and position of a free relativistic particle taken at a momentum $\tau = 0$ of proper time τ parametrizing the evolution of the particle.

Then the position of the particle in four-dimensional phase space at an arbitrary moment τ, corresponding to the Lorentz reference frame L_0 is given by $(p_0(\tau), p_1(\tau), x_0(\tau), x_1(\tau))$ where [compared with (2)]

$$p_\mu(\tau) = p_\mu^0 \tag{71a}$$

$$x_\mu(\tau) = x_\mu^0 + \frac{p_\mu^0}{M}\tau \tag{71b}$$

If we consider now an arbitrary Lorentz reference frame L moving with a velocity $v = \beta_k C$ relative to L_0 then in this reference frame the initial phase-space position is (p'_μ, x'_μ) where

$$p'_0 = \frac{p_0^0 + \beta_k p_1^0}{(1-\beta_k^2)^{1/2}} \tag{72a}$$

$$p'_1 = \frac{p_1^0 + \beta_k p_0^0}{(1-\beta_k^2)^{1/2}} \tag{72b}$$

and

$$x_0^{0\prime} = \frac{x_0^0 + \beta_k x_1^0}{(1-\beta_k^2)^{1/2}} \tag{72c}$$

$$x_1^{0\prime} = \frac{x_1^0 + \beta_k x_0^0}{(1-\beta_k^2)^{1/2}} \tag{72d}$$

So that the trajectory in phase space parametrized by τ is given in the arbitrary Lorentz frame L by

$$p'_0(\tau) = p'_0 \tag{73a}$$

$$p'_1(\tau) = p'_1 \tag{73b}$$

$$x'_0(\tau) = x_0^{0\prime} + \frac{p'_0}{M}\tau = \frac{x_0^0 + \beta_k x_1^0}{(1-\beta_k^2)^{1/2}} + \frac{p'_0}{M}\tau \tag{73c}$$

$$x'_1(\tau) = x_1^{0\prime} + \frac{p'_1}{M}\tau = \frac{x_1^0 + \beta_\kappa x_0^0}{(1-\beta_k^2)^{1/2}} + \frac{p'_1}{M}\tau \tag{73d}$$

[compare with (5) and (6)].

We suppose now that the Lorentz frame L_0 is an admissible one, so that the initial momentum (p_0^0, p_1^0) and position (x_0^0, x_1^0) of the particle can be written as

$$p_0^0 = \mu\left[\hat{p}_0^0(\hat{m} + \hat{k}_0) + \hat{k}_0\hat{p}_1^0\right] \equiv \mu\hat{p}_0 \tag{74a}$$

$$p_1^0 = \mu\left[\hat{p}_1^0(\hat{m} + \hat{k}_0) + \hat{k}_0\hat{p}_0^0\right] \equiv \mu\hat{p}_1 \tag{74b}$$

$$x_0^0 = a\left[\hat{q}_0(\hat{m} + \hat{k}_0) + \hat{k}_0\hat{q}_1\right] \equiv a\hat{x}_0 \tag{74c}$$

$$x_1^0 = a\left[\hat{q}_1(\hat{m} + \hat{k}_0) + \hat{q}_0\hat{k}_0\right] \equiv a\hat{x}_1 \tag{74d}$$

The mass of the particle M can be written as

$$M = \mu_0\hat{m} \tag{75}$$

and evolution-invariant parameter τ proper time as [see (69a)]

$$\tau = a_0\hat{m}\hat{\tau} \tag{76}$$

Then the evolution in phase space in admissible Lorentz frame L_0 is given by [see (71)]

$$p_\mu(\tau) = \mu\hat{p}_\mu \tag{77a}$$

and

$$\begin{aligned} x_0(\tau) &= a\hat{x}_0 + \frac{\mu\hat{p}_0}{\mu_0\hat{m}}a_0\hat{m}\hat{\tau} \\ &= a\left[(\hat{m} + \hat{k}_0)(\hat{q}_0 + \hat{p}_0^0\hat{\tau}) + \hat{k}_0(\hat{q}_1 + \hat{p}_1^0\hat{\tau})\right] \\ &\equiv a\left[(\hat{m} + \hat{k}_0)\hat{q}_0(\tau) + \hat{k}_0\hat{q}_1(\hat{\tau})\right] \end{aligned} \tag{77b}$$

$$\begin{aligned} x_1(\tau) &= a\hat{x}_1 + \frac{\mu\hat{p}_1}{\mu_0\hat{m}}a_0\hat{m}\hat{\tau} \\ &= a\left[(\hat{m} + \hat{k}_0)(\hat{q}_1 + \hat{p}_1^0\hat{\tau}) + \hat{k}_0(\hat{q}_0 + \hat{p}_0^0\hat{\tau})\right] \\ &\equiv a\left[(\hat{m} + \hat{k}_0)\hat{q}_1(\hat{\tau}) + \hat{k}_0\hat{q}_0(\hat{\tau})\right] \end{aligned} \tag{77c}$$

because [see (70b)]

$$\frac{\mu}{\mu_0} a_0 = a \tag{78}$$

From (77) follows that the evolution of the particle is taking place along admissible positions in phase space and that

$$\hat{q}_0(\hat{\tau}) = \hat{q}_0 + \hat{p}_0^0 \hat{\tau} \tag{79a}$$

$$\hat{q}_1(\tau) = \hat{q}_1 + \hat{p}_1^0 \hat{\tau} \tag{79b}$$

The evolution of the particle in an arbitrary Lorentz frame L given by (73) in the case when L is an admissible one is given by

$$p_0'(\tau) \equiv \mu' \hat{p}_0' = \mu'\left[\hat{p}_0^0(\hat{m} + \hat{k}) + \hat{p}_1^0 \hat{k}\right] \tag{80a}$$

$$p_1'(\tau) \equiv \mu' \hat{p}_1' = \mu'\left[\hat{p}_1^0(\hat{m} + \hat{k}) + \hat{p}_0^0 \hat{k}\right] \tag{80b}$$

$$x_0'(\tau) = a'\left[\hat{p}_0' \hat{q}_0(\hat{\tau}) + \hat{p}_1' \hat{q}_1(\hat{\tau})\right] \tag{80c}$$

$$x_1'(\tau) = a'\left[\hat{p}_0' \hat{q}_1(\hat{\tau}) + \hat{p}_1' \hat{q}_0(\hat{\tau})\right] \tag{80d}$$

where $\hat{q}_\mu(\hat{\tau})$ is given by (79).

From (80) it follows that for each admissible moment of proper time τ characterized by an integer $\hat{\tau}$, in all admissible Lorentz frames the particle occupies the admissible position.

In the nonrelativistic limit of small $\beta_k (\hat{k} \ll \hat{m})$ we have that

$$\beta_k = \frac{v_k}{c} = \frac{\hat{k}}{\hat{m}} \tag{81}$$

$$\mu_k = \left[1 - \frac{\hat{k}}{\hat{m}} + \frac{3}{2}\left(\frac{\hat{k}}{\hat{m}}\right)^2\right] \mu_0 \tag{82}$$

so that nonrelativistic energy E momentum p, position x, and time t are given by

$$E = p_0 - MC^2 = \mu_\epsilon (\hat{k}')^2, \qquad k' = \hat{k}_0 + \hat{p}_1^0 \tag{83a}$$

$$p = \mu_p \hat{k}' \tag{83b}$$

$$x = a(\hat{q}_1 + \hat{k}_0 \hat{q}_0 + \hat{k}' \hat{\tau}) \tag{83c}$$

$$t = a_t(\hat{q}_0 + \hat{\tau}) \tag{83d}$$

in complete analogy with the classical case considered in Section 2.

5. DISCRETIZATION OF THE ROTATIONS

Keeping in mind the further generalization of the discretization procedure of free-particle kinematics to the case of $3+1$ dimensions we consider the discretization of pure rotations which are subgroups of both nonrelativistic and relativistic free-particle kinematical groups. As we shall see one can put forward the discretization procedure for the case of admissible rotations which form a compact set in complete analogy to the discretization procedure which results in choosing a noncompact set of admissible Minkovsky non-Eucledian rotations considered above.

As previously we consider initially the discretization procedure of the momentum of the particle.

So let us suppose that the moment of the particle is along the x_1 axis and is equal to

$$p_1 = \mu \hat{m} \tag{84}$$

Then after rotation by the angle $\theta \leqslant \pi/2$ in the $x_1 x_2$ plane we obtain that

$$p_1' = p_1 \cos\theta = \mu \hat{m} \cos\theta \tag{85a}$$

$$p_2' = p_1 \sin\theta = \mu \hat{m} \sin\phi \tag{85b}$$

Let us rewrite this in the form

$$p_1' = \mu \frac{1+\alpha}{(1+\alpha^2)^{1/2}} \left(\frac{m}{1+\alpha} \right) \tag{86a}$$

$$p_2' = \mu \frac{1+\alpha}{(1+\alpha^2)^{1/2}} \left(\frac{\alpha m}{1+\alpha} \right), \qquad \alpha > 0 \tag{86b}$$

where $\alpha = \tan\theta$. In full analogy with the procedure of Section 3 we consider the factor

$$\mu_\theta = \mu \frac{1+\alpha}{(1+\alpha^2)^{1/2}} \tag{87}$$

as a new scale obtained after rotation.

The discretization procedure can be introduced in a natural way if we suppose that the admissible rotations obey the following conditions:

$$\frac{\hat{m}}{1+\alpha}=\hat{m}-\hat{k}_2 \tag{88a}$$

$$\frac{\hat{m}\alpha}{1+\alpha}=\hat{k}_2 \tag{88b}$$

so that

$$\alpha_{k_2}=\frac{\hat{k}_2}{\hat{m}-\hat{k}_2} \tag{89}$$

which we can compared with (56).

From (85)–(88) it follows that after rotation by an admissible angle, p_1 and p_2 components of the momentum can be written in the following form:

$$p_1'=\mu_\theta(\hat{m}-\hat{k}_2)\equiv\mu_\theta\hat{p}_1' \tag{90a}$$

$$p_2'=\mu_\theta\hat{k}\equiv\mu_\theta\hat{p}_2' \tag{90b}$$

that is as products of a scale factor and integers.

Let us mention that $\alpha=\tan\theta$, $(0\leqslant\theta\leqslant\pi/2)$ is taking all admissible values if the integer $\hat{k}_2$ is taking $\hat{m}$ admissible values

$$0\leqslant\hat{k}_2\leqslant\hat{m} \tag{91}$$

and that

$$\hat{p}_1+\hat{p}_2'=\hat{m} \tag{92}$$

is an invariant of the discrete admissible rotations in this case.

The other cases can be treated similarly (see Appendix). So we can conclude that there exist $4\hat{m}$ admissible rotation angles and correspondingly $4\hat{m}$ admissible directions of the momentum vector. This is illustrated in Figure 2.

In the same way rotations of the momentum vector leave invariant the quadratic form

$$p_1^2+p_2^2=\mu_0^2\hat{m}^2 \tag{93}$$

so that they all lie on the circle, the transformation of the integers $\hat{p}$ leave invariant the form

$$|\hat{p}_1|+|\hat{p}_2|=\hat{m} \tag{94}$$

so that their end points lie on the square (see Figure 2).

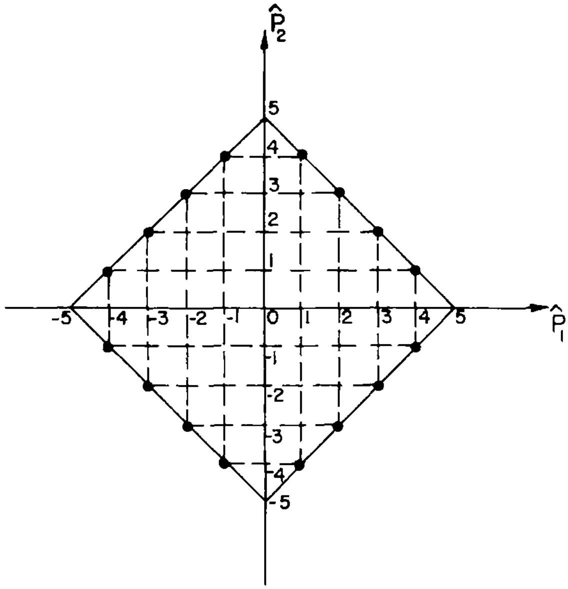

Fig. 2. The $4\hat{m}$ admissible rotation angles and $4\hat{m}$ admissible directions of the momentum vector.

Let us suppose now that in the system of reference where the momentum of the particle is along the x_1 axis direction and is equal as before to $p_1 = \mu\hat{m}$, the position of the particle is given by

$$x_1 = a\hat{m}\hat{q}_1 \tag{95a}$$

$$x_2 = a\hat{m}\hat{q}_2 \tag{95b}$$

where $\hat{q}_1$ and $\hat{q}_1$ are arbitrary integer numbers. Then the admissible rotations lead to the transformation

$$x_1' = x_1 \cos\theta + x_2 \sin\theta \tag{96a}$$

$$x_2' = x_2 \cos\theta - x_1 \sin\theta \tag{96b}$$

which can be written also as

$$x_1' = a\frac{1+\alpha}{(1+\alpha^2)^{1/2}}\left(\frac{\hat{m}}{1+\alpha}\hat{q}_1 + \frac{\hat{m}\alpha}{1+\alpha}\hat{q}_2\right) = a_\theta(\hat{p}_1'\hat{q}_1 + \hat{p}_2'\hat{q}_2) \tag{97a}$$

$$x_2' = a\frac{1+\alpha}{(1+\alpha^2)^{1/2}}\left(\frac{\hat{m}}{1+\alpha}\hat{q}_2 - \frac{\hat{m}\alpha}{1+\alpha}\hat{q}_1\right) = a_\theta(\hat{p}_2'\hat{q}_2 - \hat{p}_2'\hat{q}_1) \tag{97b}$$

So we see that admissible rotations of the position of the particle given by (97) lead to the positions for which the coordinates are of the admissible type products of the scale factors and integers.

6. LORENTZ $O(3,1)$ GROUP ON THE LATTICE

Now, after the considerations of the discretization procedures of the main elements of the Lorentz group—two-dimensional Lorentz boosts and two-dimensional rotations—we can turn to the general case and consider the discretization procedure of the Lorentz group $O(3,1)$.

The timelike momentum vector can be written in general as

$$p_0 = M\cosh w \tag{98a}$$

$$p_1 = M\sin w\cos\theta \tag{98b}$$

$$p_2 = M\sinh w\sin\theta\cos\varphi \tag{98c}$$

$$p_3 = M\sinh w\sin\theta\sin\varphi \tag{98d}$$

where M is the mass of the particle.

We suppose that $M = \mu_0 \hat{m}^3$, where μ_0 is a scale parameter with dimension of mass and $\hat{m}$ is an integral dimensionless number.

We suppose now that the boost parameter w and rotation angles θ and ϕ are admissible [see (50) and (89)] so that

$$\tanh w_{k_1} = \frac{\hat{k}_1}{\hat{m} + \hat{k}_1} \tag{99a}$$

$$\tan\theta_{k_2} = \frac{\hat{k}_2}{\hat{m} - \hat{k}_2}, \qquad \tan\varphi_{k_3} = \frac{\hat{k}_3}{\hat{m} - \hat{k}_3} \tag{99b}$$

in the case when $0 \leqslant w < \infty$ and $0 \leqslant \theta, \phi \leqslant \pi/2$ and analogous formulas for different values of (w, θ, ϕ).

Then it easy to verify that

$$p_0 \equiv \mu_0 \hat{p}_0 = \mu_0(\hat{m} + \hat{k}_1)\hat{m}^2 \tag{100a}$$

$$p_1 = \mu_1 \hat{p}_1 = m_1 \hat{k}_1(\hat{m} - \hat{k}_2)\hat{m} \tag{100b}$$

$$p_2 \equiv \mu_2 \hat{p}_2 = \mu_2 \hat{k}_1 \hat{k}_2(\hat{m} - \hat{k}_3) \tag{100c}$$

$$p_3 \equiv \mu_3 \hat{p}_3 = \mu_3 \hat{k}_1 \hat{k}_2 \hat{k}_3 \tag{100d}$$

where

$$\mu_0 = \mu\gamma(w_{k_1}) \tag{101a}$$

$$\mu_1 = \mu_0\gamma(\theta_{k_2}) \tag{101b}$$

$$\mu_2 = \mu_3 = \mu_1\gamma(\varphi_{k_3}) \tag{101c}$$

and

$$\gamma(w_{k_1}) = \frac{1-\tanh w_{k_1}}{\left(1-\tanh^2 w_{k_1}\right)^{1/2}} \tag{102a}$$

$$\gamma(\theta_{k_2}) = \frac{1+\tan\theta_{k_2}}{\left(1+\tan^2\theta_{k_2}\right)^{1/2}} \tag{102b}$$

$$\gamma(\varphi_{k_3}) = \frac{1+\tan\varphi_{k_3}}{\left(1+\tan^2\varphi_{k_3}\right)^{1/2}} \tag{102c}$$

It is easy to see that the following expression

$$\hat{p}_0 - |\hat{p}_1| - |\hat{p}_2| - |\hat{p}_3| = \hat{m}^3 \tag{103}$$

is the invariant of transformation (100).

Let us mention that when $\mu \to 0$, then $\hat{m} \to \infty$ and from (99) follows that all parameters $w_{k_1}, \theta_{k_2}, \phi_{k_3}$ fall into continuum in this limit.

We suppose now that in the system of rest of the particle when $p_\mu^0 = (M, \mathbf{0})$ the initial four-dimensional position is given by the vector $Q_\mu^0 = (a\hat{m}^3\hat{q}_0, a\hat{m}^3\hat{\mathbf{q}})$, where a is the scale parameter of coordinates in the system of rest of the particle and $\hat{q}_\mu$ some arbitrary integers. Then the transformation from the system of rest of the particle to some admissible Lorentz system where the momentum of the particle is given by (100) is accompanied by the transformation of the coordinates of the particle of the following form:

$$Q_0 = \frac{p_0}{M} Q_0^0 + \frac{1}{M}(\mathbf{p}\mathbf{Q}^0) \tag{104a}$$

$$\mathbf{Q} = \mathbf{Q}^0 + \frac{\mathbf{p}}{m}\left[\frac{(\mathbf{p}\mathbf{Q}^0)}{p_0 + M} + Q_0^0\right] \tag{104b}$$

where p_μ is given by (100), the coordinate Q_μ^0 is equal to

$$Q_\mu^0 = a\hat{m}^3\hat{q}_\mu^0 \tag{105}$$

and the mass M is equal to

$$M = \mu\hat{m}^3 \tag{106}$$

So, we see that in the admissible system both the momentum of the particle and its position are completely determined by two state factors—μ and a—and eight integers $(\hat{m}, \hat{k}_1, \hat{k}_2, \hat{k}_3, \hat{q}_\mu^0)$.

It is easy to see after rewriting (104) in the following form

$$Q_0 = a(p_0\hat{q}_0^0 + \mathbf{p}\hat{\mathbf{q}}^0) \tag{107a}$$

$$Q_1 = a\left[p_1\hat{q}_0^0 + \left(p_0 - \frac{p_2^2 + p_3^2}{p_0 + M}\right)\hat{q}_1^0 + \frac{p_1 p_2}{p_0 + M}\hat{q}_2^0 + \frac{p_1 p_3}{p_0 + M}\hat{q}_3^0\right] \tag{107b}$$

$$Q_2 = a\left[p_2\hat{q}_0^0 + \frac{p_1 p_2}{p_0 + M}\hat{q}_1^0 + \left(p_0 - \frac{p_1^2 + p_3^2}{p_0 + M}\right)\hat{q}_2^0 + \frac{p_2 p_3}{p_0 + M}\hat{q}_3^0\right] \tag{107c}$$

$$Q_3 = a\left[p_3\hat{q}_0^0 + \frac{p_1 p_3}{p_0 + M}\hat{q}_1^0 + \frac{p_2 p_3}{p_0 + M}\hat{q}_2^0 + \left(p_0 - \frac{p_1^2 + p_2^2}{p_0 + M}\right)\hat{q}_3^0\right] \tag{107d}$$

that if the spatial part of the vector of the momentum is along one axis we recover the two-dimensional case (60) considered above.

Let us mention also that the relativistic Galilean evolution of the particle parametrized by proper time parameter τ in the arbitrary admissible Lorentz frame is given by

$$x^\mu(\tau) = Q^\mu + \frac{p^\mu}{M}\tau, \qquad \text{where } \tau = a\hat{m}^3\hat{\tau} \tag{108}$$

so that taking into account (104) we have that this evolution is described by (104) where now we must consider the position variable Q_0^0 depending on τ:

$$Q_0^0(\tau) = a\hat{m}^3(\hat{q}_0 + \hat{\tau}) \tag{109}$$

Note that $\mathbf{Q}_0$ is τ independent.

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APPENDIX

Let us consider now three other regions of change of the rotation angles:

(a) $$-\pi/2 \leqslant \theta \leqslant 0$$

Then

$$\mu_\theta = \frac{1-\alpha}{(1+\alpha^2)^{1/2}}\mu = \frac{1+|\alpha|}{(1+\alpha^2)^{1/2}}\mu \tag{A.1}$$

$$\alpha = \frac{\hat{k}_2}{\hat{m}-\hat{k}_2}, \qquad 0 \leqslant k_2 \leqslant m \tag{A.2}$$

$$\hat{p}'_1 = \frac{\hat{m}}{1-\alpha} = \hat{m} - \hat{k}_2 \tag{A.3}$$

$$\hat{p}'_2 = \frac{\hat{m}\alpha}{1-\alpha} = -\hat{k}_2 \tag{A.4}$$

so that

$$\hat{p}'_1 - \hat{p}'_2 = \hat{p}'_1 + |\hat{p}'_2| = \hat{m} \tag{A.5}$$

is an invariant of the transformation in this case.

(b) $$\pi/2 \leqslant \theta \leqslant \pi$$

$$\mu_\theta = \frac{1-\alpha}{(1+\alpha^2)^{1/2}}\mu = \frac{1+|\alpha|}{(1+\alpha^2)^{1/2}}\mu \tag{A.6}$$

$$\alpha = -\frac{\hat{k}_2}{\hat{m}-\hat{k}_2} \leqslant 0, \qquad 0 \leqslant \hat{k}_2 \leqslant \hat{m} \tag{A.7}$$

$$\hat{p}'_1 = \frac{\hat{m}\alpha}{1-\alpha} = -\hat{k}_2 \tag{A.8}$$

$$\hat{p}'_2 = \frac{m}{1-\alpha} = \hat{m} - \hat{k}_2 \tag{A.9}$$

so that

$$\hat{p}'_1 - \hat{p}'_1 = |\hat{p}'_1| + |\hat{p}'_2| = \hat{m} \tag{A.10}$$

is an invariant of the transformation.

(c) $$-\pi \leqslant \theta \leqslant -\pi/2$$

$$\mu_\theta = \frac{1+\alpha}{(1+\alpha^2)^{1/2}}\mu \tag{A.11}$$

$$\alpha = \frac{\hat{k}_2}{\hat{m} - \hat{k}_2} \geqslant 0, \qquad \theta \leqslant \hat{k}_2 \leqslant \hat{m} \tag{A.12}$$

$$\hat{p}'_1 = -\frac{\hat{m}}{1+\alpha} = -(\hat{m} - \hat{k}_2) \tag{A.13}$$

$$\hat{p}'_2 = -\frac{\hat{m}\alpha}{1+\alpha} = -\hat{k}_2 \tag{A.14}$$

so that

$$|\hat{p}'_1| + |\hat{p}'_2| = \hat{m} \tag{A.15}$$

is invariant of the transformation in this case.